J Glob Optim (2007) 39:347–363
DOI 10.1007/s10898-007-9141-5

ORIGINAL PAPER

Globally optimal solutions of max–min systems

Yuegang Tao · Guo-Ping Liu · Wende Chen

Received: 9 March 2006 / Accepted: 17 January 2007 / Published online: 21 February 2007

Abstract A variety of problems in computer science, operations research, control theory, etc., can be modeled as non-linear and non-differentiable max–min systems. This paper introduces the global optimization into such systems. The criteria for the existence and uniqueness of the globally optimal solutions are established using the high matrix, optimal max-only projection set and $\mathbf{k}^s$-control vector of max–min functions. It is also shown that the global optimization can be accomplished through the partial max-only projection representation with algebraic and combinatorial features. The methods are constructive and lead to an algorithm of finding all globally optimal solutions.

Keywords Global optimization · High matrix · $\mathbf{k}^s$- control vector · Max–min system · Optimal max-only projection set

1 Introduction

A number of problems arising in computer science, operations research, control theory, etc., can be modeled as discrete-event systems with maximum and mini-

Y. Tao (✉)
Laboratory of Complex Systems and Intelligence Science, Institute of Automation, Chinese Academy of Sciences, Beijing 100080, Peoples Republic of China
e-mail: yuegangtao@yahoo.com.cn

G.-P. Liu
Department of Engineering, University of Glamorgan, Pontypridd CF37 1DL, UK

G. -P. Liu
Laboratory of Networked Control Systems, Central South University, Changsha 410083, Peoples Republic of China

W. Chen
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, Peoples Republic of China

mum constraints which include examples less well-known to mathematicians, such as digital circuits, computer networks or automated manufacturing plants (Baccelli et al. 1992; Chakraborty et al. 1999; Du and Pardalos 1995; Gunawardena 1993a, b; Hulgaard et al. 1995; Sakallan et al. 1992; Szymanski and Shenoy 1992). These systems, called max–min systems, are described by non-linear and non-differentiable max–min functions in which the operations maximum, minimum, and addition appear simultaneously. Max–min systems are non-linear extensions of max systems with only maximum constraints (or only minimum constraints). Max systems which are based on two operations maximum and addition are well understood by methods based on max-plus algebra (Baccelli et al. 1992; Cassandras and Lafortune 1999; Cohen et al. 1984, 1989; Cuninghame-Green 1979). The study of max–min systems arouses the great interest of some researchers from different fields since 1990s (Baccelli and Mairesse 1998; Cochet-Terransson et al. 1999; Gaubert and Gunawardena 1998a, b; Gunawardena 1993a, b, 1994a, b; Hulgaard et al. 1995; Olsder 1991, 1993; Sakallan et al. 1992; Szymanski and Shenoy 1992; van der Woude and Subiono 2003). The final strand in the historical sketch begins with the work of Olsder (1991) on the existence of a fixed point (or an eigenvector) in separated max–min systems. Then, there have been many studies for max–min systems, primarily on the existence and calculation of a fixed point and a cycle time (or an eigenvalue). For example, by introducing max-only and min-only projection representations of max–min functions, Gunawardena (1994b) described the Duality Conjecture which gives not only the existence of a cycle time but also a method of calculating it in terms of the structure of max–min functions. Gaubert and Gunawardena (1998a, b) proved the Duality Conjecture (usually called Duality Theorem) and provided an algorithm to compute a cycle time based on policy iteration. Cochet-Terransson et al. (1999) obtained a constructive fixed point theorem for general max–min systems. van der Woude and Subiono (2003) presented a necessary and sufficient condition of the structural existence of an eigenvalue and a corresponding fixed point for bipartite max–min systems.

There has been increasing interest in the programming of max–min systems due to its significance both in theory and applications and much closely related work has been done. For example, for the disjunctive programming, Beaumont (1990) presented an algorithm for disjunctive programming problems; Lee and Grossmann (2000, 2001) presented a global optimization algorithm for non-convex generalized disjunctive programming problems that involve bilinear, linear fractional, and concave separable functions; Grossmann and Lee (2003) proposed a convex non-linear relaxation of the non-linear convex generalized disjunctive programming problem that relies on the convex hull of each of the disjunctions, that is, obtained by variable disaggregation and reformulation of inequalities. For the minimal realization and complementarity, De Schutter and De Moor(1995a, b) showed that the minimal state space realization problem in max-plus algebra can be transformed into an extended linear complementarity problem, which is an extension of the well-known linear complementarity problem; Gaubert et al. (1998) showed that the minimal dimension of a linear realization over the (max,+) semiring of a convex sequence is equal to the minimal size of a decomposition of the sequence as a supremum of discrete affine maps; Ralph (2002) showed how the stability or conditioning properties of globally metrically regular horizontal linear complementarity problems are preserved by a homotopy framework for solving the HLCP that finds a stable direction at each iteration as a local minimizer of a strongly convex quadratic program with linear complementarity constraints; De Schutter et al. (2002) showed the extended linear complementarity problem with bounded surplus variables over the feasible set or with a bounded feasible set can be recast as a standard linear complementarity problem. For the cutting angle method, Andramonov et al. proposed the cutting angle method which is a deterministic technique is applicable to a very broad class of non-convex global optimization problems (Andramonov et al. 1999; Rubinov 2000); Batten and Beliakov described the fast algorithm for the cutting angle method of global optimization (Batten and Beliakov 2002; Rubinov 2000); Bagirov and Rubinov dealt with combinations of the cutting angle method in global optimization (Bagirov and Rubinov 2003; Rubinov 2000).

This paper considers the following programming problem for max–min systems:

$$\begin{aligned} &\text{minimize } \ F(\mathbf{x}), \\ &\text{subject to } \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{1}$$

where $F(\mathbf{x})$ is a max–min function of $\mathbb{R}^n$ to $\mathbb{R}^m$, $\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n | \sum_{j=1}^n x_j = b,\ x_j \geq 0, 1 \leq j \leq n, b > 0\}$. Problems that can be described by (1) appear in a number of practical applications with regard to scheduling and games, and belong to non-linear and non-differentiable programming (Du et al. 2001; Horst et al. 2000; Pardalos et al. 2002). It is notable that the objective function in (1) is a vector valued function.

To motivate the real-world background of the programming problem above, let us see a manufacturing system with both maximum and minimum types of timing constraints. To simplify the presentation, the following assumptions are made on this system: (1) This manufacturing system consists of five machines $M_i, 1 \leq i \leq 5$; (2) Machine M_1 starts a processing when two workpieces transported from M_3 and M_4, respectively, arrive at M_1; (3) Machine M_2 starts a processing when a workpiece transported from either M_3 or M_5 arrives at M_2; (4) The processing finish time of a workpiece at M_3, M_4, and M_5 is variable and the transport time of a workpiece between two machines is fixed; (5) The sum of the processing finish time of M_3, M_4, and M_5 is constant. Let T_1 and T_2 be the processing start time of machines M_1 and M_2, respectively, and similarly τ_3, τ_4, and τ_5 the processing finish time of machines M_3, M_4, and M_5. Also let $\delta_{1,3}$ and $\delta_{1,4}$ denote the transport time from M_3 and M_4 to M_1, respectively, and similarly $\delta_{2,3}$ and $\delta_{2,5}$ the transport time from M_3 and M_5 to M_2. So, it is clear from assumptions (2) and (3) above that the processing start time of M_1 and M_2 can be described by

$$\begin{aligned} T_1 &= \max(\delta_{1,3} + \tau_3, \delta_{1,4} + \tau_4), \\ T_2 &= \min(\delta_{2,3} + \tau_3, \delta_{2,5} + \tau_5). \end{aligned} \tag{2}$$

Let $\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$. To improve the efficiency of this manufacturing system, one needs to find the earliest start time of machines M_1 and M_2 when the total processing finish time of machines M_3, M_4, and M_5 is given. Taking assumptions (4) and (5) into account, the earliest start time of machines M_1 and M_2 can be determined by solving the following minimization problem:

$$\text{minimize } \mathbf{T} \ \text{ subject to } \tau_3 + \tau_4 + \tau_5 = \text{constant}, \tag{3}$$

where $\mathbf{T}$ is a max–min function of $\mathbb{R}^3$ to $\mathbb{R}^2$ and is a non-linear and non-differentiable vector valued function. It can be seen from (2) that the variables of components T_1 and T_2 are not independent. When the global optimal solution to the above minimization problem is obtained, the operating time of the individual machines in the system can effectively be adjusted to achieve the optimal design of the system.

The global optimization will be introduced to solve programming problem (1). It is interesting to obtain characterization of global solutions of the optimization. Based on the combinatorial nature of the problem, this paper will establish the criteria of the existence and uniqueness of the globally optimal solutions and develop an algorithm of finding all solutions. Also some new concepts on max–min functions will be introduced and the max-only projection representation of max–min functions will be used to solve the problem.

The next section describes the basic definitions of global optimization and globally optimal solution to programming problem (1). These definitions provide the main themes for the rest of the paper. Section 3 introduces and characterizes the concept of high matrix and establishes the criteria of the existence and uniqueness of globally optimal solutions of max-only systems (Theorem 1). By introducing the optimal max-only projection set and $\mathbf{k}^s$-control vector of max–min functions, Section 4 proves two main results, Theorems 2 and 3, which give the criteria of the existence and uniqueness of globally optimal solutions of max–min systems. Section 5 presents an algorithm of finding all globally optimal solutions and provides a numerical example to illustrate it. Also, this section discusses the complexity of the algorithm and shows that for max-only systems, the algorithm has polynomial bound (Theorem 4). Section 6 draws some conclusions of the paper and highlights future work.

2 Basic definitions

Let us start with the partial order on real vectors, an arbitrary max–min function and max-plus algebra. After discussing some necessary technicalities it dissects the programming problem (1) and gives the definitions of global optimization and globally optimal solution.

Let $\mathbb{R}$ be a set of all real numbers and $\mathbb{R}^n$ an n-dimensional column vector set over $\mathbb{R}$. The notations $a \vee b$ and $a \wedge b$ stand for the maximum and minimum of real numbers a and b, respectively, i.e.,

$$a \vee b := \max(a, b) \quad \text{and} \quad a \wedge b := \min(a, b).$$

It is easy to see that $+$ distributes over both $\vee$ and $\wedge$, i.e.,

$$(a \vee b) + c = (a + c) \vee (b + c) \quad \text{and} \quad (a \wedge b) + c = (a + c) \wedge (b + c).$$

It is assumed that $+$ always has higher precedence than either $\vee$ or $\wedge$ in this paper. The equalities above can hence be rewritten as

$$(a \vee b) + c = a + c \vee b + c \quad \text{and} \quad (a \wedge b) + c = a + c \wedge b + c,$$

respectively. Vectors in $\mathbb{R}^n$ are denoted by a bold lower case letter, e.g., $\mathbf{x}$ and x_j denotes the jth component of $\mathbf{x}$. The notation $\mathbf{x} \leq \mathbf{y}$ denotes the usual partial order on $\mathbb{R}^n$, i.e.,

$$\mathbf{x} \leq \mathbf{y} \Longleftrightarrow x_j \leq y_j, \quad \text{for } 1 \leq j \leq n. \tag{4}$$

The operations $\vee$ and $\wedge$ are also applied to vectors:

$$(\mathbf{x} \vee \mathbf{y})_j = x_j \vee y_j \quad \text{and} \quad (\mathbf{x} \wedge \mathbf{y})_j = x_j \wedge y_j \quad \text{for } 1 \leq j \leq n.$$

A max–min function of type $(n, 1)$ is any function $f : \mathbb{R}^n \to \mathbb{R}^1$, which can be written as a term in the following grammar:

$$f := x_1, \ldots, x_n | f + a | f \vee f | f \wedge f, \tag{5}$$

where $x_1, \ldots, x_n$ are variables and $a \in \mathbb{R}$ is a parameter, and denoted by $f(\mathbf{x})$. The notation above is the Backus–Naur form known from computer science. Here the vertical bars separate the different ways in which terms can recursively be constructed. The simplest term is one of the n variables, x_j, thought of as the jth component function. Given any term, a new one may be constructed by adding a; given two terms, a new one may be constructed by taking the maximum or the minimum. Only these rules may be used to build terms. For example, for the following three terms

$$((x_1 + 1 \wedge x_2) \vee x_3) + 1, \; x_2 \wedge 2, \; x_2 + 4 \vee x_1 + x_3,$$

the first is a max–min function which can be regarded as of type $(3, 1)$ but neither the second nor the third can be generated by the grammar (5). Although the above grammar provides a convenient syntax to write terms, this paper is interested in them only as functions from $\mathbb{R}^n$ to $\mathbb{R}$. Terms can therefore be rearranged using the distributivity of the operations $\vee$ and $\wedge$, as well as the fact that $+$ distributes over both $\vee$ and $\wedge$. The first term above can hence be simplified further to

$$(x_1 + 2 \vee x_3 + 1) \wedge (x_2 + 1 \vee x_3 + 1).$$

A max–min function of type (n, m) is any function $F : \mathbb{R}^n \to \mathbb{R}^m$, such that each component is a max–min function of type $(n, 1)$, and denoted by $F(\mathbf{x})$. It is easy to see from the definitions that $F(\mathbf{x})$ is a non-linear and non-differentiable function. A max–min function of type $(n, 1)$ which uses only $\vee$ and $+$ is said to be max-only. A max–min function of type (n, m) is said to be max-only if its components are all max-only.

A max–min system is a system that can be described using a max–min function. A max-only system, usually a max system, is a special max–min system where all the components are max-only functions.

For $h \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, set $\mathbf{x} + h = [x_1 + h \; \ldots \; x_n + h]^\tau$, where τ denotes the transpose of a vector. The following lemma collects three well-known properties of max–min functions of type (n, m) together.

Lemma 1 *Let $F(\mathbf{x})$ be a max–min function of type (n, m). If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $h \in \mathbb{R}$, then the following statements hold.*

1. *Continuity* $F(\mathbf{x})$ *is continuous in* $\mathbf{x}$.
2. *Monotonicity* $\mathbf{x} \leq \mathbf{y} \Longrightarrow F(\mathbf{x}) \leq F(\mathbf{y})$.
3. *Homogeneity* $F(\mathbf{x} + h) = F(\mathbf{x}) + h$.

Max-plus algebra is a structure consisting of the set $\mathbb{R} \cup \{-\infty\}$ together with two operators $\vee$ and $+$, denoted by $\mathbb{R}_{\max}$. The zero element for $\vee$ is $-\infty : a \vee -\infty = a$ for all $a \in \mathbb{R}_{\max}$. Usually, $-\infty$ is denoted by ε. Let $\mathbb{R}_{\max}^{m \times n}$ be an $m \times n$ matrix set over $\mathbb{R}_{\max}$. If $A = [a_{ij}] \in \mathbb{R}_{\max}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$, then the product $A\mathbf{x}$ of A and $\mathbf{x}$ over $\mathbb{R}_{\max}$ is the m-dimensional column vector whose ith component is defined as $\vee_{1 \leq j \leq n} a_{ij} + x_j$. A more complete overview of max-plus algebra can be found in Baccelli et al. (1992) and Cuninghame-Green (1979).

The component of max–min function $F(\mathbf{x})$ of type (n, m), denoted by $F_i(\mathbf{x})$, can uniquely be placed in the so-called conjunctive normal form:

$$F_i(\mathbf{x}) = f_i^1(\mathbf{x}) \wedge \ldots \wedge f_i^{l(i)}(\mathbf{x}), \tag{6}$$

where the max-only functions $f_i^{\alpha_i}(\mathbf{x}) = a_{i1}^{\alpha_i} + x_1 \vee \ldots \vee a_{in}^{\alpha_i} + x_n$, $a_{ij}^{\alpha_i} \in \mathbb{R}_{\max}, 1 \leq \alpha_i \leq l(i)$, are of type $(n, 1)$, $l(i)$ is the number of max-only functions of type $(n, 1)$ in (6), and for $1 \leq \alpha_i \neq \beta_i \leq l(i), [a_{i1}^{\alpha_i} \ \ldots \ a_{in}^{\alpha_i}] \leq [a_{i1}^{\beta_i} \ \ldots \ a_{in}^{\beta_i}]$ does not hold (Here, it is arranged that $\forall a \in \mathbb{R}_{\max}, a \geq \varepsilon$.). $a_{ij}^{\alpha_i}$ is called the coefficient of the jth variable x_j of $f_i^{\alpha_i}(\mathbf{x})$. If $a_{ij}^{\alpha_i} = \varepsilon$, a term of the form $a_{ij}^{\alpha_i} + x_j$ merely indicates the absence of the variable x_j, i.e., x_j does not contribute in $f_i^{\alpha_i}(\mathbf{x})$. Since each component $F_i(\mathbf{x})$ must have at least one variable in it, there must exist an $a_{ij}^{\alpha_i} \neq \varepsilon$ in $f_i^{\alpha_i}(\mathbf{x})$. $F_i(\mathbf{x})$ can be put in conjunctive form using simplification rules in Section 3.1.3 in Baccelli et al. (1992) and its normal form can be obtained by the way of Section 2 in Gunawardena (1994a). The conjunctive normal form (6) is unique up to re-ordering of the $f_i^{\alpha_i}(\mathbf{x})$. The proof of the uniqueness can be found in Appendix in Gunawardena (1994a). Let us continue with our example. The conjunctive normal form of the max–min function $((x_1 + 1 \wedge x_2) \vee x_3) + 1$ of type $(3, 1)$ is

$$(2 + x_1 \vee \varepsilon + x_2 \vee 1 + x_3) \wedge (\varepsilon + x_1 \vee 1 + x_2 \vee 1 + x_3).$$

Let $\mathbf{c}$ denote the m-dimensional column vector $[c_1 \ \ldots \ c_m]^\tau$, where

$$c_i = \wedge_{1 \leq \alpha_i \leq l(i)} \vee_{1 \leq j \leq n} a_{ij}^{\alpha_i}, \ \ 1 \leq i \leq m. \tag{7}$$

It follows immediately from $F_i(\mathbf{x}) \neq \varepsilon, 1 \leq i \leq m$ and (6) that $\mathbf{c} \in \mathbb{R}^m$. The next lemma gives the range of $F(\mathbf{x})$ over $\mathcal{X}$.

Lemma 2 *Let $F(\mathbf{x})$ be a max–min function of type (n, m). For any $\mathbf{x} \in \mathcal{X}$, $\mathbf{c} \leq F(\mathbf{x}) \leq \mathbf{c} + b$.*

Proof Let $\mathbf{b}$ denote the n-dimensional column vector $[b \ \ldots \ b]^\tau$. $\forall \mathbf{x} \in \mathcal{X}$, since $\mathbf{x} \geq \mathbf{0} := [0 \ \ldots \ 0]^\tau (\in \mathbb{R}^n)$, $[\sum_{j=1}^n x_j \ \ldots \ \sum_{j=1}^n x_j]^\tau \geq \mathbf{x}$, i.e., $\mathbf{b} \geq \mathbf{x}$. By the properties of monotonicity and homogeneity of $F(\mathbf{x})$,

$$\mathbf{c} = F(\mathbf{0}) \leq F(\mathbf{x}) \leq F(\mathbf{b}) = F(\mathbf{0} + b) = F(\mathbf{0}) + b = \mathbf{c} + b.$$

The proof is completed. □

Programming problem (1) can be solved by considering the problem below: Given $\mathcal{X}$ and $F(\mathbf{x})$, find at least one vector $\bar{\mathbf{x}} \in \mathcal{X}$ such that

$$F(\bar{\mathbf{x}}) \leq F(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{X} \tag{8}$$

or show that such a vector does not exist.

Definition 1 Problem (8) is called the *global optimization* of programming problem (1) and a vector $\bar{\mathbf{x}}$ is said to be the *globally optimal solution*. $F(\bar{\mathbf{x}})$ is called the *global minimum* of $F(\mathbf{x})$ over $\mathcal{X}$.

For the constraint condition $\sum_{j=1}^n x_j = b, x_j \geq 0, 1 \leq j \leq n, b > 0$, Lemma 2 indicates that $\mathbf{c}$ is the lower bound of $F(\mathbf{x})$ with respect to the partial order of (4). Based on

Lemma 2 and the property of continuity of $F(\mathbf{x})$, the solution set of (8) and that of the equation $F(\mathbf{x}) = \mathbf{c}$ are the same. Hence, the global optimization (8) can be reduced to the study of the equation $F(\mathbf{x}) = \mathbf{c}$.

Let $F(\mathbf{x})$ be a max–min function of type (n, m) in which each of the components is written as (6). If $a_{1j}^{\alpha_1} = \ldots = a_{mj}^{\alpha_m} = \varepsilon$ in the coefficients of the jth variable x_j, then by setting $\bar{x}_j = b, \bar{x}_k = 0, 1 \le k \le n, k \ne j$, the globally optimal solution $\bar{\mathbf{x}} = [\bar{x}_1 \ \ldots \ \bar{x}_n]^\tau$ is obtained. For this case, without loss of generality, in the rest of this paper, it is assumed that $F(\mathbf{x})$ satisfies the following condition:

$$[a_{1j}^{\alpha_1} \ \ldots \ a_{mj}^{\alpha_m}]^\tau \ne \theta, \quad \text{for } 1 \le \alpha_i \le l(i), \ 1 \le i \le m, \ 1 \le j \le n, \tag{9}$$

where θ is a zero vector in $\mathbb{R}_{\max}^m$ (an m-dimensional column vector set over $\mathbb{R}_{\max}$).

3 Max-only systems and high matrix

This section studies the special case of global optimization (8) in which the objective function is a max-only function of type (n, m): the global optimization of max-only systems. The concepts and results to be proposed are vital to the general case.

Let $F(\mathbf{x})$ be a max-only function of type (n, m). It is easy to see that $F(\mathbf{x})$ can uniquely be written in canonical form:

$$F_i(\mathbf{x}) = a_{i1} + x_1 \vee \ldots \vee a_{in} + x_n, \quad 1 \le i \le m, \tag{10}$$

where $a_{ij} \in \mathbb{R}_{\max}$. If $A = [a_{ij}]$ is the corresponding matrix over $\mathbb{R}_{\max}$ then, using max-plus matrix notation, $F(\mathbf{x}) = A\mathbf{x}$. It is well known that there is a one-to-one correspondence between max-only functions of type (n, m) and $m \times n$ matrices (with at least one non-zero element per row) over $\mathbb{R}_{\max}$.

Definition 2 Let A be an $m \times n$ matrix over $\mathbb{R}_{\max}$. A is said to be *high* if each of its columns contains the maximal element of a row.

Before proceeding further, it may be helpful to see an example. For the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ \varepsilon & 1 & 1 \end{bmatrix},$$

2 and 1 are the maximal elements of the first and the second rows, respectively. It follows directly from Definition 2 that A is a high matrix.

The following notations are used to characterize the high matrix. Let $A = [a_{ij}]$ be an $m \times n$ matrix over $\mathbb{R}_{\max}$. Set

$$\mathcal{Q}_j = \{q | a_{qj} \ne \varepsilon\}, \quad 1 \le j \le n, \tag{11}$$

$$k_j = \min_{q \in \mathcal{Q}_j} (c_q - a_{qj}), \quad 1 \le j \le n, \tag{12}$$

$$\mathcal{T}_i = \{t | a_{it} \ge a_{ij}, 1 \le j \le n\}, \quad 1 \le i \le m, \tag{13}$$

$\mathcal{N} = \{1, \ldots, n\}$ and $\mathbf{k} = [k_1 \ \ldots \ k_n]^\tau$. It follows from assumption (9) that $\mathcal{Q}_j \ne \emptyset, 1 \le j \le n$, where $\emptyset$ is an empty set. Two characterizations of high matrix can now easily be deduced.

Lemma 3 *The following statements are equivalent.*

1. *A is high.*
2. $\mathbf{k} = \mathbf{0}$.
3. $\cup_{1 \le i \le m} \mathcal{T}_i = \mathcal{N}$.

Proof It will be shown that each of the latter two statements are equivalent to the first.

($1 \Longleftrightarrow 2$) It is easy to see that the jth column of A contains a maximal element of a row if and only if $k_j = 0$. Hence A is a high matrix if and only if $\mathbf{k} = \mathbf{0}$.

($1 \Longleftrightarrow 3$) For any column index j, it follows from $\cup_{1 \le i \le m} \mathcal{T}_i = \mathcal{N}$ that there exists a $\mathcal{T}_{i_0}$ such that $j \in \mathcal{T}_{i_0}$. Then the element of A in the i_0th row and the jth column is a maximal element of the i_0th row, and therefore the jth column contains a maximal element of the i_0th row. Hence A is a high matrix.

Conversely, $\forall j \in \mathcal{N}$, let the jth column of A contain the maximal element $a_{i'j}$ of the row i'. It follows immediately from (13) above that $j \in \mathcal{T}_{i'}$. So, $\cup_{1 \le i \le m} \mathcal{T}_i \supseteq \mathcal{N}$. But it is clear that $\cup_{1 \le i \le m} \mathcal{T}_i \subseteq \mathcal{N}$. Hence $\cup_{1 \le i \le m} \mathcal{T}_i = \mathcal{N}$. The proof is completed. □

The next result is one of the key contributions of this paper.

Theorem 1 *Let $F(\mathbf{x})$ be a max-only function of type (n, m) and A the corresponding matrix. The global optimization* (8) *has a solution if and only if*

$$\sum_{j=1}^{n} k_j \ge b. \tag{14}$$

Furthermore, if an equality in (14) *holds then* (8) *has a unique solution.*

Proof Since (14) holds, for any decomposition $b = \sum_{j=1}^{n} b_j, b_j \ge 0$, and $b_j \le k_j, 1 \le j \le n$, by setting $\mathbf{b} := [b_1 \ \dots \ b_n]^\tau$, $\mathbf{b} \in \mathcal{X}$. If the jth column of A contains a maximal element of a row, then $k_j = 0$, and so $b_j = 0$. Hence, it follows that $F(\mathbf{b}) = \mathbf{c}$, i.e., $\mathbf{b}$ is a globally optimal solution of (8). Conversely, let $\bar{\mathbf{x}} = [\bar{x}_1 \ \dots \ \bar{x}_n]^\tau$ be a solution of the global optimization (8), i.e., $F(\bar{\mathbf{x}}) = \mathbf{c}$. By an easy direct computation, it can be obtained that

$$\bar{\mathbf{x}} \le \mathbf{k}. \tag{15}$$

Hence $b = \sum_{j=1}^{n} \bar{x}_j \le \sum_{j=1}^{n} k_j$.

Let $\mathbf{x} = [x_1 \ \dots \ x_n]^\tau$ be any solution of (8). It follows that (14) holds. It can be seen that the equality in (14) holds implies that the equality in (15) holds, i.e., $\mathbf{x} = [k_1 \ \dots \ k_n]^\tau$. The proof is completed. □

Since $\sum_{j=1}^{n} k_j \ge b > 0$, $\mathbf{k} \ne \mathbf{0}$. It follows immediately from Lemma 3 that if a max-only system has a globally optimal solution then the corresponding A is certainly a non-high matrix. If a max-only system satisfies condition (14), it has at least one globally optimal solution. It can be seen from the first half proof of Theorem 1 that if the number of decompositions of b is infinite then the global optimization of a max-only system has an infinite number of solutions. It is easy to see that if inequality (14) is strict and there are two non-zero components in $k_j, 1 \le j \le n$, then the number of decompositions of b is infinite. Hence, when there are two non-zero components in $k_j, 1 \le j \le n$, the global optimization has a unique solution that implies the equality in (14) holds.

4 Globally optimal solutions

Let us now consider a general case of global optimization (8) and begin with recalling the max-only projection representation of max–min functions.

Let $F(\mathbf{x})$ be a max–min function of type (n, m) such as (6). $[a_{i1}^{\alpha_i} \ \ldots \ a_{in}^{\alpha_i}]$ over $\mathbb{R}_{\max}$ is said to be the coefficient row vector of $f_i^{\alpha_i}(\mathbf{x})$. An $m \times n$ max-plus matrix A associating with $F(\mathbf{x})$ is constructed by taking the coefficient row vector of $f_i^{\alpha_i}(\mathbf{x})$ as the ith row of A. The matrix A constructed in this way is called a *max-only projection* of $F(\mathbf{x})$. The set of all max-only projections of $F(\mathbf{x})$ is uniquely determined and is denoted by $P(F)$. $P(F)$ is clearly with combinational nature. It follows from the conjunctive normal form (6) that the coefficient row vectors corresponding to $f_i^{\alpha_i}(\mathbf{x})$ are distinct. By the way constructing max-only projections and the multiplication principle, $P(F)$ contains $\prod_{i=1}^{m} l(i)$ distinct max-only projections, where two matrices are regarded as equal if and only if their (i, j)-entries are equal. It follows easily from (6) that any max–min function can be rewritten as

$$F(\mathbf{x}) = \wedge_{r \in \mathcal{I}} A_r \mathbf{x}, \tag{16}$$

where $A_r \in \mathbb{R}_{\max}^{m \times n}$ are max-only projections of $F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathcal{I}$ is the finite index set of all max-only projections. The equality (16) is called the *max-only projection representation* of $F(\mathbf{x})$. When $m = n$, the representation (16) is the key link in a chain of a calculation for a cycle time (Gaubert and Gunawardena 1998a; Gunawardena 1994b). The following investigation also proceeds from the representation (16).

Lemma 4 *Let $f_i^{\alpha_i}(\mathbf{x})$ be a max-only function of type $(n, 1)$ of $F_i(\mathbf{x})$ such as* (6). *If* $\max\{a_{i1}^{\alpha_i}, \ldots, a_{in}^{\alpha_i}\} > c_i$, *then $f_i^{\alpha_i}(\mathbf{x}) > c_i$ for any $\mathbf{x} \in \mathcal{X}$.*

Proof Since $\max\{a_{i1}^{\alpha_i}, \ldots, a_{in}^{\alpha_i}\} > c_i, f_i^{\alpha_i}(\mathbf{0}) > c_i$. $\forall \mathbf{x} \in \mathcal{X}$, since $\mathbf{x} \geq \mathbf{0}$, it follows from the property of monotonicity of $f_i^{\alpha_i}(\mathbf{x})$ (by Lemma 1) that $f_i^{\alpha_i}(\mathbf{x}) \geq f_i^{\alpha_i}(\mathbf{0}) > c_i$. The proof is completed. □

Based on Lemma 4, some max-only functions of type $(n, 1)$ of $F_i(\mathbf{x})$ such as (6) do not need to consider for the global optimization. The max-only function $f_i^{\alpha_i}(\mathbf{x})$ of type $(n, 1)$ of $F_i(\mathbf{x})$ is said to be *redundant* if it satisfies the condition of Lemma 4. By removing all redundant max-only functions from a conjunctive normal form of $F_i(\mathbf{x})$, the partial conjunctive normal form of $F_i(\mathbf{x})$ is obtained, which is denoted by $\bar{F}_i(\mathbf{x})$. Correspondingly, the subset of $P(F)$ and the partial representation of $F(\mathbf{x})$ can be obtained, respectively. The former is denoted by $SP(F)$, the latter is denoted by $\bar{F}(\mathbf{x}) := \wedge_{s \in \mathcal{J}} A_s \mathbf{x}$, where $\mathcal{J}$ is the index set of all max-only projections in $SP(F)$. Clearly, $\mathcal{J}$ is a subset of $\mathcal{I}$.

Definition 3 Let $F(\mathbf{x})$ be a max–min function of type (n, m). The set $SP(F)$ is called the *optimal max-only projection set* of $F(\mathbf{x})$. $\bar{F}(\mathbf{x}) := \wedge_{s \in \mathcal{J}} A_s \mathbf{x}$ is called the *partial max-only projection representation* of $F(\mathbf{x})$.

The criterion for the existence of globally optimal solutions can now be deduced below.

Theorem 2 *Global optimization* (8) *has a solution if and only if there exists at least a max-only projection satisfying inequality* (14) *in $SP(F)$.*

Proof Let $\bar{\mathbf{x}}$ be a globally optimal solution of (8). Since $F_i(\bar{\mathbf{x}}) = c_i$, for $1 \le i \le m$, there must exist the max-only functions of type $(n, 1)$, $f_i^{\alpha_i}(\mathbf{x})$, such that $f_i^{\alpha_i}(\bar{\mathbf{x}}) = c_i$. By the coefficient row vectors of $f_i^{\alpha_i}(\mathbf{x}), 1 \le i \le m$, the max-only projection A_s can be constructed and $A_s\bar{\mathbf{x}} = \mathbf{c}$. By Lemma 4, $A_s \in SP(F)$. It follows from Theorem 1 that A_s satisfies condition (14).

Conversely, let A_s be a max-only projection in $SP(F)$. Since A_s satisfies condition (14), by Theorem 1, there exists an $\tilde{\mathbf{x}}$ such that $A_s\tilde{\mathbf{x}} = \mathbf{c}$ in $\mathcal{X}$. It follows from Lemma 4 that $A_r\tilde{\mathbf{x}} \ge \mathbf{c}$ for all $A_r \in P(F)$. Hence $F(\tilde{\mathbf{x}}) = \mathbf{c}$, i.e., $\tilde{\mathbf{x}}$ is a globally optimal solution of (8). The proof is completed. □

The proof of Theorem 2 also results in the following corollary.

Corollary 1 *All solutions of* $A_s\mathbf{x} = \mathbf{c}$, *for any* $s \in \mathcal{J}$, *are solutions of* (8). *Conversely, a solution of* (8) *is a solution of* $A_{s_0}\mathbf{x} = \mathbf{c}$, *for some* $s_0 \in \mathcal{J}$.

Before proceeding to the next main result—the uniqueness of globally optimal solutions—it will be convenient and useful to introduce the following notation first.

Any max-only projection A_s in $SP(F)$ corresponds to the column vector

$$[k_1^s \ \dots \ k_n^s]^\tau, \tag{17}$$

where $k_j^s, 1 \le j \le n$, are defined as (12).

Definition 4 The column vector (17) is called the sth *control vector* of $F(\mathbf{x})$ relative to $SP(F)$ and is denoted by $\mathbf{k}^s$.

From Theorem 2 and Definition 4 the following corollary can immediately be deduced.

Corollary 2 *Global optimization* (8) *has a solution if and only if* $F(\mathbf{x})$ *has at least a control vector* $\mathbf{k}^s$ *satisfying the inequality* $\sum_{j=1}^n k_j^s \ge b$.

Theorem 3 *If all control vectors satisfying condition* (14) *are the same and satisfy the equality in* (14), *then global optimization* (8) *has a unique solution.*

Proof Let $\mathbf{k}^s$ be a control vector satisfying the inequality $\sum_{j=1}^n k_j^s \ge b$ and A_s the corresponding max-only projection. Since $\sum_{j=1}^n k_j^s = b$, by Theorem 1, $A_s\mathbf{x} = \mathbf{c}$ has a unique solution $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}} = \mathbf{k}^s$. Since all control vectors $\mathbf{k}^s$ are just the same, all the corresponding $A_s\mathbf{x} = \mathbf{c}$ have only one solution. Hence, by Corollary 1, (8) has a unique solution. The proof is completed. □

It is clear that if global optimization (8) has a unique solution then all the control vectors satisfying condition (14) are the same. Furthermore, if this same control vector contains at least two non-zero components, then it must satisfy the equality in (14).

It can be seen from Theorems 2 and 3 that the existence and uniqueness of the globally optimal solutions depend on the optimal max-only projection set $SP(F)$. In fact, Corollary 2 and Theorem 3 imply that for any system, all globally optimal solutions are determined by the control vectors. For $1 \le i \le m$, if $F_i(\mathbf{x})$ do not have any redundant max-only function, $SP(F) = P(F)$, and the latter is quite large. It is notable that the uniqueness of globally optimal solutions implies a stronger constraint on all control vectors.

5 Algorithm and example

First, based on the constructive methods in the previous sections, the following algorithm is presented to find all globally optimal solutions of (8).

Algorithm

Step 1 Distribute $+$ over $\vee$ and $\wedge$ such that there exist only terms of the form $x_j + a$ with respect to $\vee$ and $\wedge$ in $F_i(\mathbf{x})$.

Step 2 Distribute $\vee$ over $\wedge$ to obtain a conjunctive form of $F_i(\mathbf{x})$.

Step 3 For $f_i^{\beta_i}(\mathbf{x})$, if there exists an $f_i^{\alpha_i}(\mathbf{x})$ such that $a_{ij}^{\alpha_i} \le a_{ij}^{\beta_i}, 1 \le j \le n$ in a conjunctive form of $F_i(\mathbf{x})$, remove $f_i^{\beta_i}(\mathbf{x})$ to obtain the normal form of $F_i(\mathbf{x})$.

Step 4 Taking $\mathbf{x} = \mathbf{0}$ in the conjunctive normal form $F_i(\mathbf{x})$, obtain c_i.

Step 5 If $\max\{a_{i1}^{\alpha_i}, \ldots, a_{in}^{\alpha_i}\} > c_i$, remove $f_i^{\alpha_i}(\mathbf{x})$ from the conjunctive normal form of $F_i(\mathbf{x})$ to obtain $\bar{F}_i(\mathbf{x})$.

Step 6 Construct the set $SP(F)$.

Step 7 Compute $\mathbf{k}^s$ for A_s in $SP(F)$, and pick out A_s if $\mathbf{k}^s \ne \mathbf{0}$.

Step 8 If $\mathbf{k}^s$ satisfies the inequality (14), decompose $b = \sum_{j=1}^n b_j$ such that $0 \le b_j \le k_j^s, 1 \le j \le n$. Taking $\tilde{\mathbf{x}} = [b_1 \ \ldots \ b_n]^\tau$, obtain a solution of $A_s\mathbf{x} = \mathbf{c}$.

Step 9 Put the solutions of all $A_s\mathbf{x} = \mathbf{c}$ together and obtain all globally optimal solutions of (8).

Next, a numerical example is given to illustrate how the algorithm works in practice.

Example The max–min system $\mathcal{G}$ is described by the max–min function $G(\mathbf{x})$ of type $(3, 3)$, whose three components are given by

$$\begin{aligned} G_1(\mathbf{x}) &= (3 + x_3 \wedge (1 + x_1 \vee 2 + x_2 \vee x_3) + 2) \vee 4 + x_2, \\ G_2(\mathbf{x}) &= (2 + x_1 \vee 5 + x_2 \vee 1 + x_3) \wedge (3 + x_1 \vee 5 + x_2), \\ G_3(\mathbf{x}) &= (1 + x_1 \vee 3 + x_2) \wedge (x_1 \vee 2 + x_2 \vee 4 + x_3). \end{aligned} \tag{18}$$

Distribute $+$ over $\vee$, $\vee$ over $\wedge$, and apply max-plus algebra to obtain the conjunctive forms of $G_i(\mathbf{x}), 1 \le i \le 3$:

$$\begin{aligned} G_1(\mathbf{x}) &= (\varepsilon + x_1 \vee 4 + x_2 \vee 3 + x_3) \wedge (3 + x_1 \vee 4 + x_2 \vee 2 + x_3), \\ G_2(\mathbf{x}) &= (2 + x_1 \vee 5 + x_2 \vee 1 + x_3) \wedge (3 + x_1 \vee 5 + x_2 \vee \varepsilon + x_3), \\ G_3(\mathbf{x}) &= (1 + x_1 \vee 3 + x_2 \vee \varepsilon + x_3) \wedge (x_1 \vee 2 + x_2 \vee 4 + x_3). \end{aligned} \tag{19}$$

It is easy to compute $\mathbf{c} = F(\mathbf{0}) = [4 \ 5 \ 3]^\tau$. Clearly, both $G_1(\mathbf{x})$ and $G_2(\mathbf{x})$ do not have any redundant max-only function of type $(3, 1)$. Since $\max\{0, 2, 4\} > 3$, removing the redundant max-only function $f_3^2(\mathbf{x}) = x_1 \vee 2 + x_2 \vee 4 + x_3$ from the conjunctive normal form of $G_3(\mathbf{x})$, obtain $\bar{G}_3(\mathbf{x}) = 1 + x_1 \vee 3 + x_2 \vee \varepsilon + x_3$. Hence, the partial conjunctive normal forms of $G_i(\mathbf{x}), 1 \le i \le 3$ are

$$\begin{aligned} \bar{G}_1(\mathbf{x}) &= (\varepsilon + x_1 \vee 4 + x_2 \vee 3 + x_3) \wedge (3 + x_1 \vee 4 + x_2 \vee 2 + x_3), \\ \bar{G}_2(\mathbf{x}) &= (2 + x_1 \vee 5 + x_2 \vee 1 + x_3) \wedge (3 + x_1 \vee 5 + x_2 \vee \varepsilon + x_3), \\ \bar{G}_3(\mathbf{x}) &= 1 + x_1 \vee 3 + x_2 \vee \varepsilon + x_3, \end{aligned} \tag{20}$$

respectively. It is easy to see that $SP(G)$ contains four max-only projections:

$$A_1 = \begin{bmatrix} \varepsilon & 4 & 3 \\ 2 & 5 & 1 \\ 1 & 3 & \varepsilon \end{bmatrix}, \quad A_2 = \begin{bmatrix} \varepsilon & 4 & 3 \\ 3 & 5 & \varepsilon \\ 1 & 3 & \varepsilon \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 1 \\ 1 & 3 & \varepsilon \end{bmatrix}, \quad A_4 = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & \varepsilon \\ 1 & 3 & \varepsilon \end{bmatrix}.$$

By (11) and (12), the corresponding control vectors are obtained below:

$$\mathbf{k}^1 = [2\,0\,1]^\tau,\ \mathbf{k}^2 = [2\,0\,1]^\tau,\ \mathbf{k}^3 = [1\,0\,2]^\tau,\ \mathbf{k}^4 = [1\,0\,2]^\tau.$$

Hence, A_1, A_2, A_3, and A_4 are all non-high matrices.

For $\mathcal{X}^0 := \{\mathbf{x} \in \mathbb{R}^3 | \sum_{j=1}^{3} x_j = 3, x_j \geq 0, 1 \leq j \leq 3\}$, $\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3$, and $\mathbf{k}^4$ satisfy the equality in (14). Then every $A_i\mathbf{x} = \mathbf{c}$ has a unique solution, which is denoted by $\mathbf{x}_i^0$, and therefore the system $\mathcal{G}$ has only two globally optimal solutions

$$\mathbf{x}_1^0 = \mathbf{x}_2^0 = [2\,0\,1]^\tau \quad \text{and } \mathbf{x}_3^0 = \mathbf{x}_4^0 = [1\,0\,2]^\tau.$$

For $\mathcal{X}^1 := \{\mathbf{x} \in \mathbb{R}^3 | \sum_{j=1}^{3} x_j = 2, x_j \geq 0, 1 \leq j \leq 3\}$, $\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3$, and $\mathbf{k}^4$ satisfy the strict inequality in (14) and the system $\mathcal{G}$ has an infinite number of solutions, i.e.,

$$\mathbf{x}_1^1 = \mathbf{x}_2^1 = [t_1\ 0\ t_3]^\tau, \quad 0 \leq t_1 \leq 2, \quad 0 \leq t_3 \leq 1, \quad t_1 + t_3 = 2$$

and

$$\mathbf{x}_3^1 = \mathbf{x}_4^1 = [t_1\ 0\ t_3]^\tau, \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_3 \leq 2, \quad t_1 + t_3 = 2.$$

The solutions above can be rewritten together as

$$\mathbf{x}^1 = [t_1\ 0\ t_3]^\tau, \quad t_1 + t_3 = 2, \quad t_1, t_3 \geq 0.$$

Finally, let us discuss the complexity of the algorithm.

Algorithmic issue In the case of max-only systems, the method of Section 3 can give an efficient algorithm of finding all the global optimal solutions.

Theorem 4 *There is a polynomial algorithm for finding all the global optimal solutions of max-only systems.*

Proof First, it is easy to see that Steps 4, 7, and 8 in the preceding algorithm can constitute an algorithm for all the global optimal solutions of max-only systems described by (10). Next, let us show this algorithm is polynomial.

To compute c_i requires at most $n-1$ maximum operations, and then the number of the operations of finding $\mathbf{c}$ is not greater than $m(n-1)$. From (11) and (12), to compute k_i requires at most m arithmetic operations and $m-1$ minimum operations, and then the number of the operations of finding $\mathbf{k}$ is not greater than $(2m-1)n$. To compute $\sum_{j=1}^{n} k_j$ requires at most $n-1$ arithmetic operations. Therefore, the sum of operations of finding all the global optimal solutions is at most $3mn - m - 1$. Without loss of generality, suppose $m \leq n$. Hence

$$3mn - m - 1 \leq 3n^2 - n - 1.$$

The above inequality indicates that the algorithm has time complexity $O(n^2)$. The proof is completed. □

However, for general max–min systems the situation is more complex. The problem stems from the fact that a max–min function $F(\mathbf{x})$ of type (n, m) is typically presented in the form of $\bar{F}(\mathbf{x}) := \wedge_{s\in\mathcal{J}} A_s\mathbf{x}$ where $\mathcal{J}$ is a subset of $\mathcal{I}$. For the aforementioned system $\mathcal{G}$, all the globally optimal solutions can be obtained by considering only two max-only projections although $SP(G)$ contains four different max-only projections. This should imply that all global optimal solutions of (8) can be given by some of max-only projections in $SP(F)$. However, in general, to get all globally optimal solutions, the information in all the max-only projections of $SP(F)$ must be used. This situation seems very similar to rectangularity in the Duality Theorem which is known to be important in max–min systems (Cochet-Terransson et al. 1999; Gaubert and Gunawardena 1998a, b). Let us see the following specific example.

The max–min system $\mathcal{L}$ is described by the max–min function $L(\mathbf{x})$ of type $(3, 2)$, whose two components are given by

$$\begin{aligned} L_1(\mathbf{x}) &= x_1 \vee 7 + x_2, \\ L_2(\mathbf{x}) &= (5 + x_1 \vee 9 + x_2 \vee 3 + x_3) \wedge (3 + x_1 \vee 9 + x_2 \vee 5 + x_3) \\ &\quad \wedge (6 + x_1 \vee 9 + x_2 \vee 1 + x_3) \wedge (4 + x_1 \vee 9 + x_2 \vee 4 + x_3). \end{aligned} \tag{21}$$

Clearly, $SP(L) = P(L)$, and $SP(L)$ contains four max-only projections:

$$A_1 = \begin{bmatrix} 0 & 7 & \varepsilon \\ 5 & 9 & 3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 7 & \varepsilon \\ 3 & 9 & 5 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 7 & \varepsilon \\ 6 & 9 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 7 & \varepsilon \\ 4 & 9 & 4 \end{bmatrix}.$$

The corresponding control vectors

$$\mathbf{k}^1 = [4\ 0\ 6]^\tau, \quad \mathbf{k}^2 = [6\ 0\ 4]^\tau, \quad \mathbf{k}^3 = [3\ 0\ 8]^\tau, \quad \mathbf{k}^4 = [5\ 0\ 5]^\tau$$

are different. It is easy to see that for the constraint condition $\sum_{j=1}^{3} x_j = 8, x_j \geq 0, 1 \leq j \leq 3$, all globally optimal solutions of $\mathcal{L}$ must be determined by $\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3$, and $\mathbf{k}^4$.

The above analysis means that the execution time of the algorithm depends heavily on the number of max-only projections.

Therefore, the complexity of the algorithm is mainly determined by two factors: the conjunctive normal form of per component and the number of non-redundant max-only functions of type $(n, 1)$. Although the method used to obtain a conjunctive normal form are constructive in nature, its computational complexity is very high in practice. On the other hand, let $\bar{F}_i(\mathbf{x}), 1 \leq i \leq m$ consist of $l'(i)$ max-only functions of type $(n, 1)$. Then the number of max-plus projections of $\bar{F}(\mathbf{x})$ is $\prod_{i=1}^{m} l'(i)$. It is clear that $\prod_{i=1}^{m} l'(i) \leq \prod_{i=1}^{m} l(i)$. From the system $\mathcal{L}$, it is possible that an equality in the above inequality holds. Hence, searching $SP(F)$ for all max-only projections satisfying inequality (14) is prohibitively expensive.

From the above, it can be seen that the algorithm proposed in this paper is intrinsically a computationally hard problem and is not well suited for problems with a large number of variables. For such kind of systems one could try to develop algorithms that only search one solution, since in many cases all solutions are not needed. One possible approach is to find one max-only projection satisfying inequality (14).

Let us see a numerical experiment for the computation time of the algorithm. The numbers of variables are fixed to be 50–70. The results of this numerical test are shown in Table 1 below. Here n the number of variables, m the number of components, r

Table 1 Numerical results

n	m	r	s	t
50	5	3,125	500	2.6100
50	6	15,625	9,375	110.1560
60	5	3,125	3,125	19.5780
60	6	15,625	15,625	345.2350
70	5	3,125	3,125	22.7500
70	6	15,625	15,625	404.7500

the number of max-only projections in $P(F)$, s the number of max-only projections in $SP(F)$, and t the computational time in a workstation.

6 Conclusions and future work

In general, the problems of global optimization are very difficult to solve due to their combinatorial nature (Horst et al. 2000; Pardalos et al. 2002). However, it is possible to solve specially structured problems. Based on max-plus algebra and structure of max–min functions, this paper has developed the methods to solve the global optimization problem of max–min systems and established the criteria of the existence and uniqueness of globally optimal solutions. The criteria are of algebraic and combinatorial type. The proposed method is direct and constructive in nature and results in an algorithm of finding all globally optimal solutions.

It may not be necessary to obtain all solutions of the global optimization in some cases. Our further research efforts will concentrate on developing an efficient algorithm that yields only one global optimal solution. In addition, the ideas introduced in this paper may be used in the following maximization problem for max–min systems

$$\begin{aligned} &\text{maximize } F(\mathbf{x}), \\ &\text{subject to } \mathbf{x} \in \mathcal{X} \end{aligned} \tag{22}$$

and some of the concepts and results paralleling to the programming (1) can be obtained. It would be interesting to find the conditions of the existence and uniqueness of globally optimal solutions if the objective function in (1) does not satisfy the condition (9). It would also be interesting to investigate the programming problems under more constraint conditions for max–min systems (see a numerical example in Appendix). The problem discussed in this paper is also a class of non-convex global optimizations. Another direction is to develop the cutting angle method for finding all globally optimal solutions.

Acknowledgments

The authors would like to thank the anonymous referees for their comments and suggestions which improved the quality of the paper. This work was supported by the National Natural Science Foundation of China under Grants 60528002 and 60334020, and the Key Research Foundation of the Joint Laboratory of Intelligence Science and Technology, Institute of Automation of Chinese Academy of Sciences, and the University of Science and Technology of China under Grant JL0606.

Appendix: an example for more constraint conditions

This appendix provides a numerical example of the programming with more constraint conditions for max–min systems.

The max–min system $\mathcal{H}$ is described by the function $H(\mathbf{x})$ of type $(2,2)$, whose conjunctive normal forms of two components are given by

$$\begin{aligned} H_1(\mathbf{x}) &= (1 + x_1 \vee 2 + x_2) \wedge (2 + x_1 \vee 1.5 + x_2), \\ H_2(\mathbf{x}) &= (3 + x_1 \vee 1 + x_2) \wedge (2 + x_1 \vee 3 + x_2). \end{aligned} \tag{23}$$

Consider the following programming problem of the system $\mathcal{H}$:

$$\text{minimize} \;\; H(\mathbf{x}), \tag{24}$$

$$\text{subject to } g_j(\mathbf{x}) \leq 0, \quad j = 1,2, \tag{25}$$

$$0 \leq x_1, x_2 \leq 1, \tag{26}$$

$$\vee_{j=1,2}\{g_j(\mathbf{x}) \leq 0\}, \tag{27}$$

where $g_1(\mathbf{x}) = x_1 + 2x_2 - 3,\ g_2(\mathbf{x}) = 3x_1 + x_2 - 4$.

Clearly, $H(\mathbf{0}) = [2\ 3]^\tau$. Let $\mathbf{h}_0 = [2\ 3]^\tau$. For all two-dimensional column vectors $\mathbf{x}$ satisfying conditions (25) and (26), it is easy to prove from the monotonicity that $\mathbf{h}_0 \leq H(\mathbf{x})$. Let us now find a non-zero two-dimensional column vector $\mathbf{x}_0$ satisfying all constraint conditions such that $H(\mathbf{x}_0) = \mathbf{h}_0$. By the definition of max-only projection, $P(H)$ contains four max-only projections:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 1.5 \\ 3 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 2 & 1.5 \\ 2 & 3 \end{bmatrix}.$$

The programming problem above can be reduced to solve the following systems of linear equations over $\mathbb{R}_{\max}$:

$$E_1\colon A_1\mathbf{x} = \mathbf{h}_0, \quad E_2\colon A_2\mathbf{x} = \mathbf{h}_0, \quad E_3\colon A_3\mathbf{x} = \mathbf{h}_0, \quad E_4\colon A_4\mathbf{x} = \mathbf{h}_0.$$

It is clear that the equations E_1 and E_4 have no non-zero solution under constraint conditions (25) and (26). By a direct calculation, the equations E_2 and E_3 have non-zero solutions

$$\bar{\mathbf{x}}^2 = [\bar{x}_1^2\ 0]^\tau, \quad 0 < \bar{x}_1^2 \leq 1 \tag{28}$$

and

$$\bar{\mathbf{x}}^3 = [0\ \bar{x}_2^3]^\tau, \quad 0 < \bar{x}_2^3 \leq 0.5, \tag{29}$$

respectively. By comparing (28) and (29), it can be seen that $\bar{\mathbf{x}}_0 = [1\ 0]^\tau$ is a unique solution of the programming.

It can be seen from the process of the answer above that the investigation for general programming problems corresponding to this example is more difficult than one of this paper. The example suggests that some of the concepts and results of this paper should be reconsidered in the light of new results on the general programming problems. The authors believe that the high matrix, partial max-only projection, and $\mathbf{k}^s$-control vector introduced in Sections 3 and 4 will be key tools in analyzing such programming problems.

References

Andramonov, M.Y., Rubinov, A.M., Glover, B.M.: Cutting angle method in global optimization. Appl. Math. Lett. **12**(3), 95–100 (1999)

Baccelli, F., Cohen, G., Olsder, G.J., Quadrat, J.-P.: Synchronization and Linearity Wiley Series in Probability and Mathematical Statistics. Wiley, New York (1992)

Baccelli, F., Mairesse, J.: Ergodic theorem for stochastic operators and discrete event networks. In: Gunawardena, J. (ed) Idempotency. Cambridge University Press, Cambridge (1998)

Bagirov, A.M., Rubinov, A.M.: Cutting angle method and a local search. J. Glob. Optim. **27**(2, 3), 193–213 (2003)

Batten, L., Beliakov, G.: Fast algorithm for the cutting angle method of global optimization. J. Glob. Optim. **24**(2), 149–161 (2002)

Beaumont, F.: Algorithm for disjunctive programming problems. Eur. J. Oper. Res. **48**, 362–371 (1990)

Cassandras, C.G., Lafortune, S.: Introduction to Discrete Event Systems. Kluwer Academic Publishers, Boston (1999)

Chakraborty, S., Yun, K.Y., Dill, D.L.: Timing analysis of asynchronous systems using time separation of events. IEEE Trans. Comput. Aided Des. Integr. Circuits Syst. **18**, 1061–1076 (1999)

Cochet-Terransson, J., Gaubert, S., Gunawardena, J.: A constructive fixed point theorem for min-max functions. Dyn. Stability Syst. **14**(4), 407–433 (1999)

Cohen G., Moller P., Quadrat J.-P., Viot M.: Linear system theory for discrete event systems. In: Proceedings of the 23-rd Conference on Decision and Control, pp.539–544 (1984)

Cohen, G., Moller, P., Quadrat, J.-P., Viot, M.: Algebraic tools for the performance evaluation of discrete event systems. Proc. IEEE **77**(1), 39–58 (1989)

Cuninghame-Green, R.A.: Minimax Algebra. Volume 166 of Lecture Notes in Economics and Mathematical Systems. Springer, Berlin Heidelberg New York (1979)

De Schutter, B., De Moor, B.: Minimal realization in the max algebra is an extended linear complementarity problem. Syst. Control Lett. **25**(2), 103–111 (1995a)

De Schutter, B., De Moor, B.: The extended linear complementarity problem. Math. Program. **71**(3), 289–325 (1995b)

De Schutter, B., Heemels, W.P.M.H., Bemporad, A.: On the equivalence of linear complementarity problems. Oper. Res. Lett. **30**(4), 211–222 (2002)

Du, D., Pardalos, P.M. (eds): Minimax and Applications. Kluwer Academic Publishers, Dordrecht (1995)

Du, D., Pardalos, P.M., Wu, W.: Mathematical Theory of Optimization. Kluwer Academic Publishers, Dordrecht (2001)

Gaubert, S., Butkovic, P., Cuninghame-Green, R.A.: Minimal (max,+) realization of convex sequences. SIAM J. Control Optim. **36**, 137–147 (1998)

Gaubert, S., Gunawardena, J.: The duality theorem for min-max functions. C. R. Acad. Sci. **326**, 43–48 (1998a)

Gaubert, S., Gunawardena, J.: A non-linear hierarchy for discrete event systems. In: Proceedings of the Fourth Workshop on Discrete Event Systems, Cagliari, Italy (1998b)

Grossmann, I.E., Lee, S.: Generalized disjunctive programming: non-linear convex hull relaxation and algorithms. Comput. Optim. Appl. **26**, 83–100 (2003)

Gunawardena, J.: Periodic behaviour in timed systems with causality. Part I: systems of dimension 1 and 2, *Technical Report STAN-CS-93-1462*, Department of Computer Science, Stanford University (1993a)

Gunawardena, J.: Timing analysis of digital circuits and the theory of min-max functions. In: Proceedings of the TAU 93, ACM International Workshop on Timing Lssues in the Specification and Synthesis of Digital Systems (1993b)

Gunawardena, J.: Min-max functions. Discrete Event Dyn. Syst. **4**(3), 377–406 (1994a)

Gunawardena, J.: Cycle times and fixed points of min-max functions. In: Proceedings of the 11th International Conference on Analysis and Optimization of Systems, LNCIS 199, pp. 266–272. Springer, Berlin Heidelberg New York (1994b)

Horst, R., Pardalos, P.M., Thoai, N.V.: Introduction to Global Optimization-Second Edition. Kluwer Academic Publishers, Dordrecht (2000)

Hulgaard, H., Burns, S.M., Amon, T., Borriello, G.: An algorithm for exact bounds on the time separation of events in concurrent systems. IEEE Trans. Comput. **44**, 1306–1317 (1995)

Lee, S., Grossmann, I.: New algorithms for non-linear generalized disjunctive programming. Comput. Chem. Eng. **24**(9,10), 2125–2141 (2000)

Lee, S., Grossmann, I.: A global optimization algorithm for non-convex generalized disjunctive programming and applications to process systems. Comput. Chem. Eng. **25**, 1675–1697 (2001)

Olsder, G.J.: Eigenvalues of dynamic max–min systems. Discrete Event Dyn. Syst. **1**, 177–207 (1991)

Olsder, G.J.: Analysis of min-max systems. Rapports de Recherche 1904, INRIA (1993)

Pardalos, P.M., Migdalas, A., Burkard, R.: Combinatorial and Global Optimization. World Scientific Publishing Company, Singapore (2002)

Ralph, D.: A stable homotopy approach to horizontal linear complementarity problems. Control Cybern. **31**(3), 575–600 (2002)

Rubinov, A.M.: Abstract Convexity and Global Optimization. Kluwer Academic Publishers, Boston (2000)

Sakallan, K.A., Mudge, T.N., Olukotun, O.A.: Analysis and design of latchcontrolled synchronous digital circuits. IEEE Trans. Comput. Aided Des. Integr. Circuits Syst. **3**, 322–333 (1992)

Szymanski T., Shenoy N.: Verifying clock schedules. In: Digest of Technical Paper of the IEEE International Conference on Computer Aided Design of Integrated Circuit, pp. 124–131. IEEE Computer Society (1992)

van der Woude, J., Subiono: Conditions for structural existence of an eigenvalue of bipartite (min, max, +)-System. Theor. Comput. Sci. **293**, 13–24 (2003)